Resit Exam — Analysis (WPMA14004)

Thursday 7 July 2016, 9.00h–12.00h

University of Groningen

Instructions

1. The use of calculators, books, or notes is not allowed.
2. Provide clear arguments for all your answers: only answering "yes", "no", or "42" is not sufficient. You may use all theorems and statements in the book, but you should clearly indicate which of them you are using.
3. The total score for all questions equals 90. If p is the number of marks then the exam grade is $G = 1 + p/10$.

Problem 1 (6 + 4 + 5 points)

(a) Prove that $\sqrt{3}$ is irrational.

(b) Show that for each $n \in \mathbb{N}$ there exists a number $a_n \in \mathbb{Q}$ such that

$$\sqrt{3} - \frac{1}{n} < a_n < \sqrt{3}.$$

(c) Explain that $\mathbb{Q}$ does *not* satisfy the Axiom of Completeness.

Problem 2 (4 + 4 + 2 + 5 points)

Let $0 < c < 1$ and assume that the sequence (x_n) satisfies

$$|x_{n+2} - x_{n+1}| \leq c|x_{n+1} - x_n| \quad \text{for all} \quad n \in \mathbb{N}.$$

Prove the following statements:

(a) $|x_{n+2} - x_{n+1}| \leq c^n|x_2 - x_1|$ for all $n \in \mathbb{N}$.

(b) $|x_m - x_n| \leq (c^{m-2} + c^{m-3} + \cdots + c^{n-1})|x_2 - x_1|$ for all $m > n$.

(c) $|x_m - x_n| \leq \dfrac{c^{n-1}}{1-c}|x_2 - x_1|$ for all $m > n$.

(d) (x_n) is convergent.

Problem 3 (7 + 8 points)

Let $K \subset \mathbb{R}$ be a compact set. Consider the set

$$A = \big\{x \in \mathbb{R} : \text{there exists } y \in K \text{ such that } |x - y| \leq 1\big\}.$$

Prove the following statements:

(a) If (x_n) is a convergent sequence in A with $x = \lim x_n$, then $x \in A$.

Hint: there exists a sequence (y_n) in K such that $-1 \leq x_n - y_n \leq 1$ for all $n \in \mathbb{N}$.

(b) A is compact.

Problem 4 (4 + 5 + 6 points)

(a) State the Mean Value Theorem.

(b) Let $f, g : [0, \infty) \to \mathbb{R}$ be differentiable functions and assume that

$$f(0) = g(0) \qquad \text{and} \qquad f'(x) \le g'(x) \quad \text{for all} \quad x \ge 0.$$

Prove that $f(x) \le g(x)$ for all $x \ge 0$. Hint: consider the function $h(x) = f(x) - g(x)$.

(c) Prove that $x - \frac{1}{2}x^2 \le \ln(1 + x) \le x$ for all $x \ge 0$.

Problem 5 (4 + 4 + 7 points)

Consider the following series:

$$f(x) = \sum_{n=1}^{\infty} \frac{1}{\sin^2(nx) + n^2}$$

Prove the following statements:

(a) The series converges uniformly on $\mathbb{R}$.

(b) f is continuous on $\mathbb{R}$.

(c) f is differentiable on $\mathbb{R}$.

Problem 6 (7 + 4 + 4 points)

Consider the function $h : [0, 2] \to \mathbb{R}$ defined by

$$h(x) = \begin{cases} 0 & \text{if } 0 \le x < 1, \\ 1 & \text{if } x = 1, \\ 2 & \text{if } 1 < x \le 2. \end{cases}$$

(a) Prove that h is integrable on $[0, 2]$.

Define the function

$$H : [0, 2] \to \mathbb{R}, \qquad H(x) = \int_0^x h(t)dt.$$

(b) Compute $H'(x)$ for $x \ne 1$.

(c) Is H differentiable at $x = 1$? If so, compute $H'(1)$.

End of test (90 points)

Solution of Problem 1 (6 + 4 + 5 points)

(a) Assume that $\sqrt{3}$ is rational. Then there exist integers p and q such that

$$\sqrt{3} = \frac{p}{q} \quad \Leftrightarrow \quad 3q^2 = p^2.$$

We may assume that p and q do not have any common factors.
(2 points)

Since the left hand side is a multiple of 3, it follows that p^2 is a multiple of 3. This is the case if and only if p itself is a multiple of 3, so $p = 3k$. for some integer k. Therefore, we have

$$3q^2 = (3k)^2 \quad \Leftrightarrow \quad q^2 = 3k^2.$$

Since the right hand side is a multiple of 3, q must be a multiple of 3 as well.
(2 points)

Therefore, we conclude that p and q have a factor 3 in common. This contradicts our assumption that the fraction p/q is written in lowest terms. Hence, $\sqrt{3}$ is irrational.
(2 points)

(b) The rational numbers are dense in the real numbers. This means that for all $a, b \in \mathbb{R}$ with $a < b$ there exists a number $r \in \mathbb{Q}$ such that $a < r < b$.
(3 points)

Applying this with $a = \sqrt{3} - 1/n$ and $b = \sqrt{3}$ gives the desired statement.
(1 point)

(c) Consider the set $A = \{a_n : n \in \mathbb{N}\}$ where the a_n are as in part (b). Then $A \subset \mathbb{Q}$ and A is bounded above. Indeed, $a_n < \sqrt{3}$ for all $n \in \mathbb{N}$.
(2 points)

However, A does not have a least upper bound in $\mathbb{Q}$. From part (b) it follows that $\sup A = \sqrt{3}$, but from part (a) it follows that $\sqrt{3} \notin \mathbb{Q}$. So not all sets in $\mathbb{Q}$ that are bounded above have a least upper bound in $\mathbb{Q}$. This means that the Axiom of Completeness does not hold for $\mathbb{Q}$.
(3 points)

Problem 2 (4 + 4 + 2 + 5 points)

(a) For $n = 1$ the given property of (x_n) reads as

$$|x_3 - x_2| \leq c|x_2 - x_1|$$

which proves the desired statement for $n = 1$.
(1 point)

Now assume that the statement is true for some $n \in \mathbb{N}$, then

$$|x_{n+3} - x_{n+2}| \leq c|x_{n+2} - x_{n+1}| \leq c \cdot c^n |x_2 - x_1| = c^{n+1}|x_2 - x_1|,$$

which shows that the statement is also true for $n + 1$. By induction, the statement holds for all $n \in \mathbb{N}$.
(3 points)

(b) If $m > n$, then the triangle inequality gives

$$\begin{aligned} |x_m - x_n| &= |x_m - x_{m-1} + x_{m-1} - x_{m-2} + \cdots + x_{n+1} - x_n| \\ &\leq |x_m - x_{m-1}| + |x_{m-1} - x_{m-2}| + \cdots + |x_{n+1} - x_n|. \end{aligned}$$

(2 points)

Using part (a) then gives

$$|x_m - x_n| \leq c^{m-2}|x_2 - x_1| + c^{m-3}|x_2 - x_1| + \cdots + c^{n-1}|x_2 - x_1|$$

which completes the proof.
(2 points)

(c) This follows directly from

$$c^{m-2} + c^{m-3} + \cdots + c^{n-1} < \sum_{k=n-1}^{\infty} c^k = c^{n-1} \sum_{k=0}^{\infty} c^k = \frac{c^{n-1}}{1-c}.$$

(2 points)

(d) Since $0 < c < 1$ we have $\lim c^{n-1} = 0$. Hence, for all $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that

$$n \geq N \quad \Rightarrow \quad 0 < c^{n-1} < \frac{1-c}{|x_2 - x_1|}\epsilon$$

(3 points)

In particular, this gives

$$m > n \geq N \quad \Rightarrow \quad |x_m - x_n| < \epsilon$$

which shows that (x_n) is a Cauchy sequence. Since Cauchy sequences are convergent the proof is complete.
(2 points)

Note: if $|x_2 - x_1| = 0$ then this argument does not work, but in this case we have that (x_n) is a constant sequence and hence trivially convergent.

Solution of Problem 3 (7 + 8 points)

(a) Let (x_n) be a convergent sequence in A and let $x = \lim x_n$. For each $n \in \mathbb{N}$ there exists an element $y_n \in K$ such that $|x_n - y_n| \leq 1$, or, equivalently, $-1 \leq x_n - y_n \leq 1$.
(2 points)

Since K is compact the sequence (y_n) has a convergent subsequence (y_{n_k}) with $y = \lim y_{n_k} \in K$.
(2 points)

Note that $\lim x_{n_k} = x$, and $-1 \leq x_{n_k} - y_{n_k} \leq 1$ for all $k \in \mathbb{N}$. By the Order Limit Theorem it follows that $-1 \leq x - y \leq 1$, or, equivalently, $|x - y| \leq 1$. Since $y \in K$ it follows that $x \in A$.
(3 points)

(b) From part (a) it follows that A is closed, see Theorem 3.2.8.
(2 points)

Now we prove that A is bounded as well. First note that K is compact and hence bounded. This means that there exists $M > 0$ such that $|y| \leq M$ for all $y \in K$.
(2 points)

Let $x \in A$ be arbitrary, then there exists $y \in K$ such that $|x - y| \leq 1$ which implies that

$$|x| = |x - y + y| \leq |x - y| + |y| \leq 1 + M.$$

This shows that A is bounded.
(2 points)

Since A is closed and bounded it follows that A is compact.
(2 points)

Solution of Problem 4 (4 + 5 + 6 points)

(a) If $f : [a, b] \to \mathbb{R}$ is continuous on $[a, b]$ and differentiable on (a, b), then there exists a point $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

(4 points)

(b) Let $x > 0$ be fixed. The function $h = f - g$ is continuous on $[0, x]$ and differentiable on $(0, x)$ by the Algebraic Continuity and Differentiability Theorems. Therefore, we can apply the Mean Value Theorem. There exists a point $c \in (0, x)$ such that

$$h(x) - h(0) = h'(c)(x - 0).$$

(2 points)

Since $h(0) = f(0) - g(0) = 0$ and $h'(c) \leq 0$ it follows that $h(x) \leq 0$, or, equivalently, $f(x) \leq g(x)$, which completes the proof.
(3 points)

(c) Let $f(x) = x - \frac{1}{2}x^2$ and $g(x) = \ln(1 + x)$. Note that $f(0) = g(0)$ and for $x \geq 0$ we have

$$1 - x^2 \leq 1 \quad \Rightarrow \quad (1 - x)(1 + x) \leq 1 \quad \Rightarrow \quad 1 - x \leq \frac{1}{1 + x} \quad \Rightarrow \quad f'(x) \leq g'(x).$$

From part (b) it then follows that $f(x) \leq g(x)$ for all $x \geq 0$.
(3 points)

Next, let $f(x) = \ln(1 + x)$ and $g(x) = x$. Note that $f(0) = g(0)$ and for $x \geq 0$ we have

$$\frac{1}{1 + x} \leq 1 \quad \Rightarrow \quad f'(x) \leq g'(x).$$

From part (b) it then follows that $f(x) \leq g(x)$ for all $x \geq 0$.
(3 points)

Solution of Problem 5 (4 + 4 + 7 points)

(a) For all $n \in \mathbb{N}$ we define

$$f_n(x) = \frac{1}{\sin^2(nx) + n^2}.$$

Note that $|f_n(x)| \leq 1/n^2$ for all $x \in \mathbb{R}$.
(1 point)

Since the series $\sum_{n=1}^{\infty} 1/n^2$ converges, it follows from the Weierstrass M-test with $M_n = 1/n^2$ that the series $\sum_{n=1}^{\infty} f_n(x)$ converges uniformly on $\mathbb{R}$.
(3 points)

(b) Each f_n is continuous on $\mathbb{R}$: this follows from the fact that the sine function is continuous and the Algebraic Continuity Theorem. (Note that the denominator of $f_n(x)$ is never zero.)
(3 points)

Since the series $\sum_{n=1}^{\infty} f_n(x)$ converges uniformly on $\mathbb{R}$ it follows that the sum is also continuous on $\mathbb{R}$.
(1 point)

(c) Let $c > 0$ be arbitrary. The chain rule gives

$$f_n'(x) = -\frac{2n\sin(nx)\cos(nx)}{(\sin^2(nx) + n^2)^2}.$$

which implies that

$$|f_n'(x)| \leq \frac{2n}{(\sin^2(nx) + n^2)^2} \leq \frac{2n}{n^4} = \frac{2}{n^3} \quad \text{for all } x \in [-c, c].$$

Applying the Weierstrass M-test with $M_n = 2/n^3$ we find that the series $\sum_{n=1}^{\infty} f_n'(x)$ converges uniformly on $[-c, c]$.
(3 points)

Finally, note that $\sum_{n=1}^{\infty} f_n(0) = \sum_{n=1}^{\infty} 1/n^2$ converges.
(1 point)

According to Theorem 6.4.3 these two conditions imply that the function f is differentiable on $[-c, c]$. Since c is arbitrary, it follows that f is differentiable on $\mathbb{R}$.
(3 points)

Solution of Problem 6 (7 + 4 + 4 points)

Consider the function $h : [0, 2] \to \mathbb{R}$ defined by

$$h(x) = \begin{cases} 0 & \text{if } 0 \le x < 1, \\ 1 & \text{if } x = 1, \\ 2 & \text{if } 1 < x \le 2. \end{cases}$$

(a) Let $\epsilon > 0$ be arbitrary and take the partition $P = \{0, 1-\frac{1}{8}\epsilon, 1+\frac{1}{8}\epsilon, 2\}$. Then we have:

$$M_1 = \sup\{h(x) \,:\, x \in [0, 1 - \tfrac{1}{8}\epsilon]\} = 0$$
$$M_2 = \sup\{h(x) \,:\, x \in [1 - \tfrac{1}{8}\epsilon, 1 + \tfrac{1}{8}\epsilon]\} = 2$$
$$M_3 = \sup\{h(x) \,:\, x \in [1 + \tfrac{1}{8}\epsilon, 2]\} = 2$$

$$m_1 = \inf\{h(x) \,:\, x \in [0, 1 - \tfrac{1}{8}\epsilon]\} = 0$$
$$m_2 = \inf\{h(x) \,:\, x \in [1 - \tfrac{1}{8}\epsilon, 1 + \tfrac{1}{8}\epsilon]\} = 0$$
$$m_3 = \inf\{h(x) \,:\, x \in [1 + \tfrac{1}{8}\epsilon, 2]\} = 2$$

$$U(h, P) - L(f, P) = (M_1 - m_1)(1 - \tfrac{1}{8}\epsilon) + (M_2 - m_2)\tfrac{1}{4}\epsilon + (M_3 - m_3)(1 - \tfrac{1}{8}\epsilon) = \tfrac{1}{2}\epsilon$$

(5 points for a correct computation)

We conclude that for any $\epsilon > 0$ there exists a partition of $[0, 2]$ for which the difference between upper and lower sum is less than ϵ. This implies that h is integrable on $[0, 2]$.
(2 points for the conclusion)

(b) Since h is constant on the intervals $[0, 1)$ and $(1, 2]$, it follows that h is continuous on those intervals. Hence, we may apply the Fundamental Theorem of Calculus.
(2 points)

For $x \in [0, 1)$ we get $H'(x) = h(x) = 0$.
(1 point)

For $x \in (1, 2]$ we get $H'(x) = h(x) = 2$.
(1 point)

(c) If H were differentiable at $x = 1$ then H would be differentiable on the interval $[0, 2]$. Now let $0 < \alpha < 2$, then by Darboux's Theorem it follows that there exists a point $c \in [0, 2]$ such that $H'(c) = \alpha$. In view of part (b) the only possibility is $c = 1$, which would mean that $H'(1)$ takes all values between 0 and 2. This is clearly impossible, and therefore we conclude that H is *not* differentiable at $x = 1$.
(4 points)

Alternative proof. One can also show that for $x \neq 1$ we have

$$\frac{H(x) - H(1)}{x - 1} = \begin{cases} 0 & \text{if } x < 1 \\ 2 & \text{if } x > 1 \end{cases},$$

which immediately shows that the limit

$$\lim_{x \to 1} \frac{H(x) - H(1)}{x - 1}$$

does not exist. Hence, H is not differentiable at $x = 1$.